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2003 J. Phys. A: Math. Gen. 36 9875

(http://iopscience.iop.org/0305-4470/36/38/305)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 9875–9883 PII: S0305-4470(03)62732-0

Solving the dispersionless Dym equation by hodograph transformations

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Received 22 April 2003
Published 10 September 2003
Online at stacks.iop.org/JPhysA/36/9875

Abstract
The method of hodograph transformation is used to find a class of exact solutions of the dispersionless Dym (dDym) equation and its reductions. In particular, we discuss hodograph solutions of one-variable and two-variable reductions of the dDym system and investigate their associated Lax as well as Hamiltonian formalism.

PACS number: 02.30.Ik

1. Introduction

One of important issues in the studies of dispersionless integrable systems (see e.g. [1] and references therein) is to find or to solve their finite-dimensional reductions since their integrable structures have been revealed in theoretical physics and mathematics, such as 2D topological field theories (TFT), matrix models, Frobenius manifolds and conformal maps, etc (see e.g. [2–7]). In [8, 9] Kodama and Gibbons found exact solutions of the dispersionless Kadomtsev–Petviashvili (dKP) equation and its reductions by using hodograph transformations [10] and obtained general hodograph equations for hydrodynamic-type equations [11]. Then it was further pointed out [3, 6] that the hodograph equation associated with a finite-dimensional reduction of dKP is identified as the string equation [2] of the corresponding TFT so that the parameters characterizing the hierarchy flows are related to the coupling constants in TFT. In this paper, motivated by the dKP, we make an attempt to address this correspondence for the dispersionless Dym (dDym) hierarchy which, with respect to the dKP, is the nonstandard dispersionless hierarchy within the Sato approach. In [12] the Poisson structures of the dDym hierarchy were given and have been used to obtain those Poisson structures of finite-dimensional reductions of the dDym system by the Dirac procedure [13, 14]. Furthermore, in our previous work [14], we showed that a two-variable reduction of the dDym system can be described as a topological Landau–Ginzburg model with two primary fields. Therefore, inspired by these observations, it is quite interesting to investigate the solution structure of the dDym system from the point of view of hodograph transformation. Before going further, let us recall the construction of dDym theory as a dispersionless (quasi-classical) limit of the Dym theory.

2. Dispersionless Dym equations

The Dym hierarchy is defined by the Lax operator [15, 16]

$$L = u_1\partial + u_0 + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots \qquad \partial = \partial/\partial x \tag{2.1}$$

which satisfies the Lax equations

$$\partial_n L = [B_n, L] \qquad n = 2, 3, \ldots \tag{2.2}$$

where $\partial_n = \partial_{t_n}$ and $B_n = (L^n)_{\geqslant 2}$ means the differential part of order $\geqslant 2$ of L^n. The first nontrivial equation is the Dym equation in 2 + 1 dimensions [17]

$$u_t = \frac{1}{4}u^3 u_{xxx} + \frac{3}{4}u^{-1}\left[u^2\partial_x^{-1}\left(\frac{u_y}{u^2}\right)\right]_y \qquad (y = t_2, t = t_3). \tag{2.3}$$

The Lax equation (2.2) is equivalent to the following compatible linear systems

$$L\Psi_{\rm Dym} = \lambda\Psi_{\rm Dym} \qquad \partial_n\Psi_{\rm Dym} = B_n\Psi_{\rm Dym} \tag{2.4}$$

where $\Psi_{\rm Dym}$ is the so-called Baker–Akhiezer (BA) function of the system. For the dispersionless limit one can use averaging procedures, by simply taking $t_n \to \epsilon t_n = T_n$ in the linear system (2.4). It turns out that

$$L_\epsilon = u_1(T/\epsilon)(\epsilon\partial) + u_0(T/\epsilon) + u_{-1}(T/\epsilon)(\epsilon\partial)^{-1} + u_{-2}(T/\epsilon)(\epsilon\partial)^{-2} + \cdots$$

where we assume $u_{-n}(T/\epsilon) = U_{-n}(T) + O(\epsilon)$, $n \geqslant -1$. One then takes a Wentzel–Kramers–Brillouin (WKB) form for the BA function $\Psi_{\rm Dym}$

$$\Psi_{\rm Dym} = \exp\left[\frac{1}{\epsilon}S_{\rm Dym}(T, \lambda)\right].$$

Now, replacing ∂_n by $\epsilon\partial/\partial T_n$ and denoting the momentum function $P = \partial_X S_{\rm Dym}$, then $\epsilon^i\partial^i\Psi_{\rm Dym} \to P^i\Psi_{\rm Dym}$ as $\epsilon \to 0$ and the linear equation $L\Psi_{\rm Dym} = \lambda\Psi_{\rm Dym}$ implies that

$$\lambda = \sum_{n=-1}^{\infty} U_{-n}(T)P^{-n}.$$

Similarly, we apply the WKB analysis for the linear system $\partial_n\Psi_{\rm Dym} = B_n\Psi_{\rm Dym}$ which yields

$$\partial_{T_n}S_{\rm Dym} = \mathcal{B}_n = (\lambda^n)_{\geqslant 2} \tag{2.5}$$

where the subscript ($\geqslant 2$) refers to the projection of powers of P. The first few of them are

$$\begin{aligned}
&\mathcal{B}_1 = 0 \qquad \mathcal{B}_2 = U_1^2P^2 \qquad \mathcal{B}_3 = U_1^3P^3 + 3U_1^2U_0P^2\\
&\mathcal{B}_4 = U_1^4P^4 + 4U_1^3U_0P^3 + \left(6U_1^2U_0^2 + 4U_1^3U_{-1}\right)P^2\\
&\cdots.
\end{aligned}$$

Differentiating both sides of (2.5) with respect to X, we obtain the conservation equations for the momentum function P:

$$\partial_{T_n}P = \partial_X\mathcal{B}_n. \tag{2.6}$$

It is easy to show [3] that equation (2.6) can be written in Lax form as

$$\partial_n\lambda = \{\mathcal{B}_n, \lambda\} \tag{2.7}$$

where the Poisson bracket {, } is defined by

$$\{f(X,P),g(X,P)\}=\frac{\partial f}{\partial P}\frac{\partial g}{\partial X}-\frac{\partial f}{\partial X}\frac{\partial g}{\partial P}.$$

Equation (2.7) defines what we call the dDym hierarchy and determines the evolution equations for U_i. Note that $\partial_1\lambda=0$ and hence U_i do not depend on t_1. On the other hand, the compatibility equations for (2.6) (or (2.7)), i.e. $\partial^2 P/\partial T_n\partial T_m=\partial^2 P/\partial T_m\partial T_n$ (or $\partial^2\lambda/\partial T_n\partial T_m=\partial^2\lambda/\partial T_m\partial T_n$), imply the zero curvature condition:

$$\partial_{T_m}\mathcal{B}_n-\partial_{T_n}\mathcal{B}_m+\{\mathcal{B}_n,\mathcal{B}_m\}=0.$$

For $m=2$ and $n=3$ we have

$$2U_1^2U_{0X}=U_{1Y}\qquad 3\big(U_1^2U_0\big)_Y=-2U_1U_{1T}$$

where $T_2=Y, T_3=T$. Eliminating U_0 we obtain the (2+1)-dimensional dDym equation

$$U_{1T}=\frac{3}{4}U_1^{-1}\left[U_1^2\partial_X^{-1}\left(\frac{U_{1Y}}{U_1^2}\right)\right]_Y \tag{2.8}$$

which is just the dispersionless limit of (2.3) by dropping the dispersion term u_{xxx}. Motivated by the dKP theory, we would like to solve (2.8) by using hodograph transformation [8, 10]. From the T_2- and T_3-flow of the conservation equations (2.6) we have

$$P_Y=(U^2P^2)_X\qquad P_T=(U^3P^3)_X+3(U^2VP^2)_X \tag{2.9}$$

where $U_1=U, U_0=V$. Following [8], one can consider the N-reductions of (2.6) or the Lax equations (2.7) so that the momentum function P or λ defined above depends only on a set of functions $(W_1,\ldots,W_N)$ with $W_1=U$, and $(W_1,\ldots,W_N)$ satisfy commuting flows

$$\frac{\partial W}{\partial T_n}=A_n(W)\frac{\partial W}{\partial X}\qquad n\geqslant 2 \tag{2.10}$$

where the $N\times N$ matrices A_n are functions of $(W_1,\ldots,W_N)$ only. To discuss solutions of the (2+1)-dimensional dDym equation (2.8) and its reductions, we only need to consider the first two flows, i.e.,

$$W_{i,Y}=\sum_{j=1}^N A_{ij}W_{j,X}\qquad W_{i,T}=\sum_{j=1}^N B_{ij}W_{j,X}\qquad i=1,\ldots,N \tag{2.11}$$

where $A=A_2$ and $B=A_3$. Equations (2.9) together with (2.11) provide the starting point for hodograph transformations. We will comment on the higher T_n-flows ($n>3$) at the end of the paper. In the following, we shall consider the cases for $N=1$ and $N=2$ and leave the general situations for a future publication.

3. $N=1$

In this case $P=P(U)$ and (2.11) becomes

$$U_Y=A(U)U_X\qquad U_T=B(U)U_X \tag{3.1}$$

which together with (2.9) imply that

$$\begin{aligned}&(A-2U^2P)\frac{\mathrm{d}P}{\mathrm{d}U}=2UP^2\\&(B-3U^2P^2-6U^2VP)\frac{\mathrm{d}P}{\mathrm{d}U}=3U^2P^3+6UVP^2+3U^2\frac{\mathrm{d}V}{\mathrm{d}U}P^2\end{aligned} \tag{3.2}$$

with

$$A = 2U^2 \frac{\mathrm{d}V}{\mathrm{d}U} \qquad B = 3VA + \frac{3}{4}U^{-1}A^2. \tag{3.3}$$

As a consistency check, by substituting (3.3) into (3.1), we get (2.8) as expected. The solutions of (3.1), and hence that of (2.8), can be obtained by using the hodograph transformations with the change of variables $(X, Y, T) \to (U, Y, T)$ with $X = X(U, Y, T)$. The hodograph equations for X are given by

$$\frac{\partial X}{\partial Y} = -A \qquad \frac{\partial X}{\partial T} = -B = -3VA - \frac{3}{4}U^{-1}A^2$$

which can be easily integrated as

$$X + A(U)Y + \left(3VA(U) + \tfrac{3}{4}U^{-1}A^2(U)\right)T = F(U) \tag{3.4}$$

where $F(U)$ is an arbitrary function of U. Thus, in view of (3.4), the initial condition $U(X, Y, 0) = U_0(X, Y)$ for the dDym equation (2.8) can be transformed to $X + A(U_0)Y = F(U_0)$. To illustrate, let us give a concrete example. For $A(U) = -1$ we can integrate (3.2) for $P(U)$ and write down its corresponding hodograph equations. It turns out that

$$U^2P^2 + P = \lambda^2 \qquad X - Y - \tfrac{3}{4}TU^{-1} = F(U) \tag{3.5}$$

where λ is determined by the condition $P^2 + P = \lambda^2$ at $U = 1$. The first equation of (3.5) defines a one-variable reduction of the dDym hierarchy with Lax operator

$$L = \lambda^2 = \bar{U}P^2 + P \qquad \bar{U} = U^2 \tag{3.6}$$

which satisfies the Lax equations

$$\partial_n L = \{(L^{n/2})_{\geqslant 2}, L\} \qquad n = 2, 3, 5, \ldots. \tag{3.7}$$

We list the first few nontrivial equations as

$$\begin{aligned} &\partial_{T_2}\bar{U} = -\bar{U}_X && \partial_{T_3}\bar{U} = -\tfrac{3}{4}\bar{U}^{-1/2}\bar{U}_X \\ &\partial_{T_5}\bar{U} = \tfrac{5}{32}\bar{U}^{-3/2}\bar{U}_X && \partial_{T_7}\bar{U} = -\tfrac{21}{512}\bar{U}^{-5/2}\bar{U}_X \\ &\ldots \end{aligned} \tag{3.8}$$

where the T_2-flow shows that $\bar{U}$ depends on X and T_2 only through the combination X–T_2. The inversion of (3.6) gives

$$P(\bar{U}) = \frac{1}{2\bar{U}}(-1 + \sqrt{1 + 4\bar{U}\lambda^2})$$

which has asymptotic expansion in λ

$$P = \frac{\lambda}{\sqrt{\bar{U}}} - \frac{1}{2\bar{U}} - \sum_{i=0}^{\infty} \frac{1}{2}H_{2i+1}\lambda^{-(2i+1)}.$$

Here the coefficients H_i are nothing but the conserved densities associated with the Lax flows (3.7) [18], and can be expressed in terms of L as

$$H_{2i+1} = \frac{2}{2i+1}\mathrm{res}\left(L^{\frac{2i+1}{2}}\right) \qquad i = 0, 1, 2, \ldots.$$

Some of them are given by

$$H_1 = -\tfrac{1}{4}\bar{U}^{-3/2} \qquad H_3 = \tfrac{1}{64}\bar{U}^{-5/2} \qquad H_5 = -\tfrac{1}{512}\bar{U}^{-7/2}.$$

We can also rewrite the Lax flows in bi-Hamiltonian form

$$\partial_{T_{2n+1}}\bar{U} = -\frac{32n(n+1)}{(2n-1)(2n+3)}J_2\nabla H_{2n+1} = \frac{128n(n+2)}{(2n-1)(2n+5)}J_1\nabla H_{2n+3}$$

where $n = 0, 1, 2, \ldots$ and the Hamiltonian operators are defined by

$$J_2 = \bar{U}^2 \partial_X \bar{U}^2 \qquad J_1 = \bar{U}^{5/2} \partial_X \bar{U}^{5/2}.$$

We remark that the solution of the first two equations of (3.8), i.e., T_2- and T_3-flows, also satisfies the dDym equation (2.8). To find solutions for (3.8) we substitute, for example, $F(U) = U^{-1}$ and $F(U) = -U$ into the hodograph equation (3.5) and obtain

$$U(X, Y, T) = \frac{\frac{3}{4}T + 1}{X - Y} \qquad F(U) = U^{-1}$$
$$U(X, Y, T) = -\frac{1}{2}(X - Y + \sqrt{(X - Y)^2 + 3T}) \qquad F(U) = -U.$$

The first solution has the singularity defined by $X = Y$ while the second one is globally defined for $T > 0$.

In fact, we may consider other functions of the form $A(U) = \alpha U^m$, $m \in Z$. We list them below without going into details:

$$(UP)^{2-m} + \frac{\alpha(2-m)}{2(m-1)} P^{1-m} = \lambda^{2-m} \qquad (m \leqslant 0)$$
$$\frac{(UP)^{m-1}}{UP + \frac{\alpha(2-m)}{2(m-1)} U^{m-1}} = \lambda^{m-2} \qquad (m \geqslant 3)$$
$$P\,\mathrm{e}^{-2UP/\alpha} = \lambda \qquad (m = 1)$$
$$UP\,\mathrm{e}^{\alpha/2P} = \lambda \qquad (m = 2)$$

where the constant λ is fixed at $U = 1$. The associated hodograph equations describing T_2- and T_3-flows are

$$X + \alpha U^m Y + \frac{3\alpha^2(m+1)}{4(m-1)} U^{2m-1} T = F(U) \qquad (m \neq 1)$$
$$X + \alpha U Y + \frac{3}{4}\alpha^2 U(1 + 2\ln U)T = F(U) \qquad (m = 1).$$

The cases shown above provide one-variable reductions of the dDym system including the one we study in detail (i.e., $\alpha = -1$, $m = 0$). If we properly choose $F(U) \propto U^n$ with $n \in Z$ then the associated hodograph equations become algebraic equations that may be easily solved. Even for the case of $m = 1$, if we choose $F(U) = C$ or $F(U) = CU$ then the hodograph solutions can be expressed in terms of Lambert's W-function. Also one can read off the conserved densities from the expansion of P in λ in each case.

4. $N = 2$

In this case we denote $W_1 = U$, $W_2 = W$, then $P = P(U, W)$ and (2.11) has the form

$$\frac{\partial}{\partial Y}\begin{pmatrix} U \\ W \end{pmatrix} = A\frac{\partial}{\partial X}\begin{pmatrix} U \\ W \end{pmatrix} \qquad \frac{\partial}{\partial T}\begin{pmatrix} U \\ W \end{pmatrix} = B\frac{\partial}{\partial X}\begin{pmatrix} U \\ W \end{pmatrix} \tag{4.1}$$

where $A = (A_{ij})$ and $B = (B_{ij})$ are 2×2 matrix functions of U and W. By requiring that U_X and W_X are independent, (2.9) gives the equations for $P(U, W)$,

$$\begin{aligned} &(A_{11} - 2U^2 P)P_U + A_{21} P_W = 2UP^2 \\ &A_{12} P_U + (A_{22} - 2U^2 P)P_W = 0 \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&(B_{11} - 3U^3P^2 - 6U^2VP)P_U + B_{21}P_W = 3U^2P^3 + 6UVP^2 + 3U^2V_UP^2 \\
&B_{12}P_U + (B_{22} - 3U^3P^2 - 6U^2VP)P_W = 3U^2V_WP^2
\end{aligned}
\tag{4.3}
$$

where we denote $P_U = \partial P/\partial U$, $P_W = \partial P/\partial W$ and so on. From (4.2) and (4.3), we see that A and B satisfy

$$
A_{11} = 2U^2V_U \qquad A_{12} = 2U^2V_W \qquad B = 3VA + \tfrac{3}{4}U^{-1}A^2. \tag{4.4}
$$

Also, the compatibility condition for (4.1) requires A to satisfy

$$
\begin{pmatrix} -(U^{-1}\det A)_W \\ (U^{-1}\det A)_U \end{pmatrix} = A \begin{pmatrix} -(4V + U^{-1}\operatorname{Tr} A)_W \\ (4V + U^{-1}\operatorname{Tr} A)_U \end{pmatrix} \tag{4.5}
$$

where the formula $A^2 = (\operatorname{Tr} A)A - (\det A)I$ has been used. To solve (4.1), we use the hodograph transformation by changing the independent variables (X, Y, T) to (U, W, T) with the dependent variables $X = X(U, W, T)$ and $Y = Y(U, W, T)$. In terms of the new variables, (4.1) becomes

$$
\begin{pmatrix} -X_W \\ X_U \end{pmatrix} = A \begin{pmatrix} Y_W \\ -Y_U \end{pmatrix} \qquad \begin{pmatrix} \partial(X, Y)/\partial(W, T) \\ -\partial(X, Y)/\partial(U, T) \end{pmatrix} = B \begin{pmatrix} Y_W \\ -Y_U \end{pmatrix} \tag{4.6}
$$

where $\partial(X, Y)/\partial(W, T) = X_WY_T - X_TY_W$. With the relation of A and B, it is easy to show that (4.6) has solutions of the form

$$
\begin{aligned}
&X - \tfrac{3}{4}U^{-1}(\det A)\,T = F(U, W) \\
&Y + \big(3V + \tfrac{3}{4}U^{-1}\operatorname{Tr} A\big)T = G(U, W)
\end{aligned}
\tag{4.7}
$$

where we have required that X_U and Y_U (or X_W and Y_W) are independent. Also, using (4.5), (4.7), and the first equation in (4.6) one can show that F and G satisfy the linear equations

$$
\begin{pmatrix} -F_W \\ F_U \end{pmatrix} = A \begin{pmatrix} G_W \\ -G_U \end{pmatrix}. \tag{4.8}
$$

Note that, by (4.7), the associated initial condition $U(X, Y, 0) = U_0(X, Y)$ and $W(X, Y, 0) = W_0(X, Y)$ satisfying the Y equations in (4.1) can be transformed to $F(U_0, W_0) = X$ and $G(U_0, W_0) = Y$. Using (4.8) and the compatibility condition $F_{UW} = F_{WU}$ we can obtain a defining equation for $G(U, W)$. A simple example is given by the matrix

$$
A = \begin{pmatrix} 0 & 2U^2 \\ 2U & 0 \end{pmatrix}
$$

which implies $V = W$ and (4.2) can be easily solved as

$$
UP + W + \frac{1}{P} = \lambda \tag{4.9}
$$

with condition $P + 1/P = \lambda$ at $U = 1$, $W = 0$. We note that (4.9) defines a two-variable Lax reduction of the dDym system [13, 14] and the first few nontrivial Lax equations are given by

$$
\begin{aligned}
&\begin{pmatrix} U \\ W \end{pmatrix}_{T_2} = \begin{pmatrix} 2U^2W_X \\ (U^2)_X \end{pmatrix} \\
&\begin{pmatrix} U \\ W \end{pmatrix}_{T_3} = \begin{pmatrix} 3U^2(U + W^2)_X \\ 3(U^2W)_X \end{pmatrix} \\
&\begin{pmatrix} U \\ W \end{pmatrix}_{T_4} = \begin{pmatrix} 4U^2(3UW + W^3)_X \\ (4U^3 + 6U^2W^2)_X \end{pmatrix}.
\end{aligned}
\tag{4.10}
$$

The associated conserved density of the reduced system can be extracted from

$$P(U,W)=\frac{1}{2U}(\lambda-W+\sqrt{(\lambda-W)^2-4U}).$$

Expanding P in an asymptotic form

$$P=\frac{\lambda-W}{U}-\frac{1}{\lambda}-\sum_{i=1}^{\infty}H_{i+1}\lambda^{-(i+1)}$$

we obtain the conserved densities $H_i=\mathrm{res}(L^i)/i$ and some of them are given by

$$H_2=W \qquad H_3=U+W^2 \qquad H_4=3UW+W^3$$

and so on. In fact these symmetries can be expressed in bi-Hamiltonian form

$$\begin{pmatrix}U\\W\end{pmatrix}_{T_n}=J_1\nabla H_{n+1}=J_2\nabla H_n \qquad n\geqslant 2$$

where $\nabla H_n={}^t(\partial H_n/\partial U,\partial H_n/\partial W)$ and J_1 and J_2 are defined by [13, 14]

$$J_1=\begin{pmatrix}0 & U^2\partial_X\\ \partial_X U^2 & 0\end{pmatrix} \qquad J_2=\begin{pmatrix}U\partial_X U^2+U^2\partial_X U & U^2W^{-1}\partial_X W^2\\ W^2\partial_X U^2W^{-1} & 2U\partial_X U\end{pmatrix}.$$

On the other hand, the hodograph equation (4.7) is given by

$$X+3U^2T=F(U,W) \qquad Y+3WT=G(U,W)$$

where G satisfies the defining equation

$$G_{WW}-UG_{UU}-2G_U=0 \tag{4.11}$$

which, for example, has a simple solution $G=-\frac{1}{2}U^{-1}=Y+3WT$ and hence, by (4.8), $F=W=X+3U^2T$. By eliminating W, we get a hodograph equation

$$18T^2U^3+2(3XT+Y)U+1=0.$$

The above equation has a simple solution

$$U(X,Y,T)=\frac{f}{18T}-\frac{2(3XT+Y)}{3Tf}$$

with

$$f=(-162T+6\sqrt{3}\sqrt{432X^3T^3+432X^2YT^2+144XY^2T+16Y^3+243T^2})^{1/3}$$

which is a kind of shock wave (due to a multi-valued function) and satisfies the first two equations of (4.10) and the dDym equation (2.8). The second matrix A satisfying (4.5) is given by

$$A=\begin{pmatrix}-\frac{4}{3}WU^{-1} & \frac{2}{3}\\ -2U & 0\end{pmatrix}$$

which implies $V=W/3U^2$ and (4.2) can be easily solved as

$$U^3P^3+WP^2+P=\lambda^3 \tag{4.12}$$

with condition $P^3+P=\lambda^3$ at $U=1$, $W=0$. The hodograph equation (4.7) is given by

$$X-T=F(U,W) \qquad Y=G(U,W)$$

where G satisfies the defining equation

$$U^2G_{UU}+3U^3G_{WW}+2UWG_{UW}-2WG_W=0.$$

Both (4.9) and (4.12) are also two-variable Lax reductions of the Dym system that have been introduced recently in [13]. In fact, using (4.4) and (4.5), we have

$$A = \begin{pmatrix} \frac{2(1-m)}{m} W U^{2-m} & \frac{2}{m} U^{3-m} \\ 2(2-m)U & 0 \end{pmatrix} \tag{4.13}$$

provided that $V = WU^{1-m}/m, m \in Z \setminus \{0, 2\}$ and $A_{22} = 0$. Equation (4.13) describes a class of two-variable reductions of the dDym hierarchy including the previous examples ($m = 1, 3$). Integrating (4.2) with respect to U and W, respectively, we obtain

$$U^m P^m + W P^{m-1} + P^{m-2} = \lambda^m$$

where λ is fixed at $U = 1$, $W = 0$. The associated hodograph equations then are given by

$$X + \frac{3(2-m)}{m} U^{3-m} T = F(U, W) \qquad Y + \frac{3(3-m)}{2m} U^{1-m} W T = G(U, W)$$

where G satisfies the defining equation

$$U^2 G_{UU} + m(m-2)U^m G_{WW} + (m-1)UWG_{UW} + (3-m)UG_U \\ - (m-1)(m-2)WG_W = 0.$$

5. Conclusions

We have investigated exact solutions of the (2+1)-dimensional dDym equation. The conservation equations together with the hydrodynamic-type equations enable us to solve dDym equation by hodograph transformations. We discuss one-variable and two-variable reductions of the dDym system and obtain their hodograph solutions. We show that these Lax reductions possess an infinite number of symmetries and can be described in the Hamiltonian formalism. Our results not only provide exact solutions of the dDym equation but also bring out several finite-dimensional reductions of the dDym system that have never been discussed before.

Finally, we remark that it is not hard to extend hodograph equations of N-reductions to higher hierarchy flows by decomposing the $N \times N$ matrices A_n in the higher commuting flows of the hydrodynamic-type equations (2.10) into the first N independent flows [9, 10]. In fact, it can be shown that $A_n = v_n(A_2/2U^2)$ where $v_n(P)$ is a polynomial in P of order $n-1$ defined by $\partial \mathcal{B}_n/\partial P$. Due to the recursion relation between $v_n(P)$, one can show that $A_{N+k} = \sum_{i=1}^{N} \mu_k^i A_i, k \geqslant 1$ where $A_1 \equiv I_{N\times N}$ and $\mu_k^i = \mu_k^i(W)$ are scalar functions of $\{W_i\}$. This fact together with (2.10) implies that the solution of N-reduction dDym hierarchies can be given in the form $W_i(X, T_2, \ldots) = W_i^0(T_1^0, T_2^0, \ldots, T_N^0)$ where $W_i^0(X, T_2, \ldots, T_N) = W_i(X, T_2, \ldots, T_N, 0, \ldots)$ and $T_i^0 = T_i + \sum_{k=1}^{\infty} \mu_k^i T_{N+k}, 1 \leqslant i \leqslant N$ (cf [9]). Hence, the matrices $A = A_2$ associated with reductions presented previously would be a good starting point to do that. On the other hand, it has been shown [1] that solutions of dispersionless hierarchies can be constructed by the Riemann–Hilbert decomposition which has the advantage of studying symmetries of solutions. Work in these directions is now in progress.

Acknowledgment

This work is supported by the National Science Council of Taiwan under grant no NSC 91-2112-M-194-008.

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